# Linear Algebra I

**30/10/2026, Friday, 15:00 – 17:00**

## 1 Null space

$8 + 12 = 20$ pts

Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 3 & 0 & 3 & 3 \end{bmatrix}.$$

(a) Compute the reduced row echelon form of $A$. Hence determine the rank of $A$.

(b) Find a basis for the null space $N(A)$.

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REQUIRED KNOWLEDGE: Row reduction, rank, and null spaces.

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SOLUTION:

**(a)** Applying elementary row operations gives

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 3 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{\substack{\textcircled{2} \leftarrow \textcircled{2} - 2 \cdot \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - 3 \cdot \textcircled{1}}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -3 & 3 \\ 0 & 3 & -3 & 3 \end{bmatrix}.$$

Next,

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -3 & 3 \\ 0 & 3 & -3 & 3 \end{bmatrix} \xrightarrow{\substack{\textcircled{2} \leftarrow \frac{1}{3} \cdot \textcircled{2} \\ \textcircled{3} \leftarrow \textcircled{3} - 3 \cdot \textcircled{2}}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally,

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} + 1 \cdot \textcircled{2}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\operatorname{rank}(A) = 2$.

**(b)** Let $\boldsymbol{u} = (u_1, u_2, u_3, u_4)^T$. The reduced system is

$$u_1 + u_3 + u_4 = 0, \qquad u_2 - u_3 + u_4 = 0.$$

Putting $u_3 = s$ and $u_4 = t$, we obtain

$$\boldsymbol{u} = \begin{bmatrix} -s - t \\ s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$.

## 2 Characteristic polynomial

$2+1+1+2+1+3+3+2+3+3+4=25$ pts

Let $M \in \mathbb{R}^{4\times 4}$ have characteristic polynomial $p_M(\lambda) = (\lambda^2 - a^2)(\lambda^2 + a^2)$, where $a \in \mathbb{R}$. Justify all your answers.

(a) Determine the eigenvalues of $M$.

(b) Determine the determinant of $M$.

(c) Determine the trace of $M$.

(d) For which values of $a$ is $M$ nonsingular?

(e) What is the rank of $M$ when $a \neq 0$?

(f) What can you say about the rank of $M$ when $a = 0$?

(g) Is $M$ diagonalizable when $a \neq 0$?

(h) Suppose that $a = 0$. Give an example of $M$ that is not diagonalizable.

(i) Can $M$ be symmetric?

(j) Can $M$ be skew-symmetric ($M^T = -M$)?

(k) Show that $M^{30102026} = a^{30102024}M^2$.

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Required Knowledge: Characteristic polynomials, eigenvalues, diagonalizability, and the Cayley–Hamilton theorem.

SOLUTION:

(a) The eigenvalues are the roots of $p_M$, namely $a, -a, ia, -ia$. When $a = 0$, all four eigenvalues are zero.

(b) Since the determinant is the product of the eigenvalues, $\det(M) = -a^4$.

(c) The trace is the sum of the eigenvalues, so it is zero.

(d) The matrix is nonsingular precisely when $\det(M) \neq 0$; hence, precisely when $a \neq 0$.

(e) If $a \neq 0$, then $M$ is nonsingular. Thus $\operatorname{rank}(M) = 4$.

(f) If $a = 0$, then $p_M(\lambda) = \lambda^4$, so $M$ is singular. Thus $\operatorname{rank}(M) \leqslant 3$. Its precise rank cannot be determined from the characteristic polynomial alone.

(g) If $a \neq 0$, the four roots $a, -a, ia, -ia$ are distinct. Therefore $M$ is diagonalizable.

(h) For example,

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has characteristic polynomial $\lambda^4$ but it is not diagonalizable.

(i) Symmetric matrices have only real eigenvalues. If $a \neq 0$, $M$ cannot be symmetric since $ia$ and $-ia$ are nonreal. If $a = 0$, we cannot conclude whether $M$ is symmetric or not: the zero matrix is a symmetric but $M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not.

(j) Skew-symmetric matrices have only imaginary eigenvalues. If $a \neq 0$, $M$ cannot be skew-symmetric, because it has the nonzero real eigenvalues $a$ and $-a$. If $a = 0$, we cannot conclude whether $M$ is skew-symmetric or not: the zero matrix is a skew-symmetric but $M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not.

(k) By the Cayley–Hamilton theorem, $M^4 - a^4 I = 0$. Hence $M^{4k+2} = a^{4k} M^2$ for every nonnegative integer $k$. Since $30102026 = 4 \cdot 7525506 + 2$, it follows that

$$M^{30102026} = a^{30102024} M^2.$$

## 3 Polynomials of matrices 20 pts

Let $A \in \mathbb{C}^{n\times n}$ and let $p(\lambda) = \lambda^k + p_{k-1}\lambda^{k-1} + \cdots + p_0$ be a polynomial with $k \geqslant 1$. Show that $p(A)$ is nonsingular if and only if none of the roots of $p$ is an eigenvalue of $A$.

REQUIRED KNOWLEDGE: Factorization of polynomials over $\mathbb{C}$, eigenvalues, and determinants.

SOLUTION:

Since $p$ has degree $k \geqslant 1$, we can write

$$p(\lambda) = \prod_{i=1}^{k}(\lambda - \mu_i),$$

where $\mu_1, \ldots, \mu_k$ are the roots of $p$, repeated according to their multiplicities. Consequently,

$$p(A) = \prod_{i=1}^{k}(A - \mu_i I).$$

**Proof based on determinants:** Since the determinant of product of square matrices equals the product of the determinants of those matrices, we have $\det p(A) = \prod_{j=1}^{k} \det(A - \mu_j I)$. Since a matrix is nonsingular if and only if its determinant is nonzero, we see that $p(A)$ is nonsingular if and only if $\det(A - \mu_i I)$ is nonzero. The latter is equivalent to saying that none of the roots of $p$ is an eigenvalue of $A$.

**Proof without using determinants:** To prove the 'if' part, note that the matrix $A - \mu_i I$ is nonsingular for each $i$ since $\mu_i$ is not an eigenvalue of $A$. As the product of nonsigular matrices is nonsingular, we can conclude that $p(A)$ is nonsingular. For the 'only if' part, suppose that $p(A)$ is nonsingular. Let $\boldsymbol{x}$ be such that $(A - \mu I)\boldsymbol{x} = \mathbf{0}_n$ for some $\mu$ that is a root of $p$. Since $A - aI$ and $A - bI$ commute for every $a, b \in \mathbb{C}$, we have $p(A)\boldsymbol{x} = \mathbf{0}_n$. It follows from nonsingularity of $p(A)$ that $\boldsymbol{x} = \mathbf{0}_n$. Hence, $\mu$ is not an eigenvalue of $A$.

## 4 Unitary diagonalization — $5 + (10 + 10) = 25$ pts

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ be linearly independent vectors. Define $M \in \mathbb{R}^{3\times 3}$ by

$$M = \boldsymbol{x}\boldsymbol{x}^T + \boldsymbol{y}\boldsymbol{y}^T + \boldsymbol{z}\boldsymbol{z}^T.$$

(a) Is $M$ unitarily diagonalizable? Justify your answer.

(b) Suppose, in addition, that

$$\boldsymbol{x}^T\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{y} = \boldsymbol{z}^T\boldsymbol{z} = 2, \qquad \boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{z} = \boldsymbol{y}^T\boldsymbol{z} = 1.$$

(a) Find all eigenvalues of $M$ and corresponding eigenvectors.
HINT: Explain why every eigenvector has the form $\boldsymbol{v} = a\boldsymbol{x} + b\boldsymbol{y} + c\boldsymbol{z}$ and determine the eigenvalues and eigenvectors from $M\boldsymbol{v} = \lambda\boldsymbol{v}$.

(b) Find a unitary diagonalizer for $M$.

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REQUIRED KNOWLEDGE: Symmetric matrices, orthogonal diagonalization, and eigenvalues and eigenvectors.

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SOLUTION:

**(a)** We have

$$M^T = (\boldsymbol{x}\boldsymbol{x}^T + \boldsymbol{y}\boldsymbol{y}^T + \boldsymbol{z}\boldsymbol{z}^T)^T = M.$$

Thus $M$ is real symmetric. It is therefore orthogonally diagonalizable and hence unitarily diagonalizable.

**(b)(i)** Since $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are linearly independent vectors in $\mathbb{R}^3$, they form a basis. Hence every eigenvector has the form

$$\boldsymbol{v} = a\boldsymbol{x} + b\boldsymbol{y} + c\boldsymbol{z}.$$

Using the given scalar products, we obtain

$$M\boldsymbol{v} = (2a + b + c)\boldsymbol{x} + (a + 2b + c)\boldsymbol{y} + (a + b + 2c)\boldsymbol{z}.$$

Thus $M\boldsymbol{v} = \lambda\boldsymbol{v}$ if and only if

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Therefore, the eigenvalues of $M$ are precisely the eigenvalues of

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Now $B$ has eigenvalue 4 with eigenvector $(1, 1, 1)^T$, and eigenvalue 1 with the two linearly independent eigenvectors $(1, -1, 0)^T$ and $(0, 1, -1)^T$. Hence the eigenvalues of $M$ are $4, 1, 1$, with corresponding eigenvectors

$$\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}, \qquad \boldsymbol{x} - \boldsymbol{y}, \qquad \boldsymbol{y} - \boldsymbol{z},$$

respectively.

**(b)(ii)** We use Gram–Schmidt on the two eigenvectors

$$\boldsymbol{u}_1 = \boldsymbol{x} - \boldsymbol{y}, \qquad \boldsymbol{u}_2 = \boldsymbol{y} - \boldsymbol{z}$$

belonging to the eigenvalue 1. First,

$$\boldsymbol{q}_2 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = \frac{\boldsymbol{x} - \boldsymbol{y}}{\sqrt{2}}.$$

Moreover, $\boldsymbol{u}_2^T \boldsymbol{q}_2 = -1/\sqrt{2}$, and therefore

$$\widetilde{\boldsymbol{u}}_2 = \boldsymbol{u}_2 - (\boldsymbol{u}_2^T \boldsymbol{q}_2)\boldsymbol{q}_2 = \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y} - 2\boldsymbol{z}).$$

Thus

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{u}}_2}{\|\widetilde{\boldsymbol{u}}_2\|} = \frac{\boldsymbol{x} + \boldsymbol{y} - 2\boldsymbol{z}}{\sqrt{6}}.$$

For the eigenvalue 4, let

$$\boldsymbol{q}_1 = \frac{\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}}{2\sqrt{3}}.$$

Thus, with

$$Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the matrix $Q$ is orthogonal (and hence unitary), and

$$Q^T M Q = D.$$

Therefore $Q$ is a unitary diagonalizer of $M$.

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10 pts free